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The Localization Properties of a Random Steady Flow on a Lattice

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We consider a random stationary vector field on a multidimensional lattice and investigate flow-connected subsets of the lattice invariant under the action of the associated flow. The subsets of primary interest are cycles, and vortices each of which is the set of orbits terminating in the same cycle. We prove that with probability 1 each vortex only involves a finite number of sites of the lattice. Under the assumption of independence of the vector field in different sites, we find that with probability 1 the vortices exhaust all possible maximal flow-connected invariant subsets of the lattice if and only if the probability of existence of a cycle is positive. Thus, if cycles exist, a particle under the action of the flow only moves within a bounded region, i.e., it is completely localized.

KEY WORDS: Multidimensional lattice; random flow; localization.

INTRODUCTION

The dynamics of a particle (classical or quantum) in a random environment has been the object of intensive investigations for many years. One of the first remarkable results in this field is due to Anderson,[1] who found the physical mechanism of the localization of an electron provided by the randomness of the environment. The rigorous mathematical formulation of Anderson localization in the multidimensional case was given quite recently (see ref. 2 and references therein). The idea of localization as a natural feature of disordered systems was developed and extended to various types of waves (see ref. 3 and references therein). Nevertheless, accurate mathematical treatment and interpretation of localization for each type of random medium is a difficult problem. For instance, the recent research on

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822/66/5-6-27

random walks in the lattice $\mathbb{Z}^d$ in a random environment (see refs. 4 and 5 and references) has shown that the diffusive character of the random walk holds if the rate of randomness in the environment is sufficiently small.

In this paper we consider the motion of a classical particle in a random steady flow on the lattice $\mathbb{Z}^d$. Our primary interest focuses on the localization properties of the flow. Namely, we investigate whether a particle under the action of the flow typically moves within a bounded region, whether the maximal flow-connected subsets invariant under the action of the flow are finite, and how big these subsets are.

Analyzing the structure of a statistically homogeneous random flow, we found that the orbits of the flow and their sets can be classified in a simple manner. Namely, there are the following objects of interest: (i) cycles, i.e., periodic orbits of the flow; (ii) vortices, each of which is the set of orbits terminating in the same cycle (thus, it is a maximal flow-connected invariant subset); and (iii) regular maximally extended orbits that may start at some site and pass through each site not more than once.

Theorem 3 states that with probability 1 each vortex only involves a finite number of sites of the lattice. Furthermore, if the vector field associated with the flow takes on independent values at different sites, the classification of the subsets invariant under the action of the flow becomes very simple. Namely, the result of Theorem 4 is that with probability 1 either there are no vortices at all or the set of the sites associated with them exhausts the entire lattice. Thus, if cycles exist with positive probability, then the lattice can be partitioned into finite regions, each of which is a maximal flow-connected invariant set containing exactly one cycle of the flow, i.e., a complete localization takes place. In particular, a particle moving under the action of such a flow will be eventually trapped in a cycle. Finally, in Theorem 6 we obtain statistical estimates of the space occupied by a vortex.

1. STATEMENT OF RESULTS

Let $\mathbb{Z}^d$ be the d-dimensional lattice, and $v(x)$, $x \in \mathbb{Z}^d$, be a random stationary vector field with values in $\mathbb{Z}^d$, i.e., $v(x) \in \mathbb{Z}^d$. That is, if we denote the probabilistic space by $(\Omega, \mathscr{F}, \mathbf{P})$, there exist a group of automorphisms preserving this probabilistic space, T_x, $x \in \mathbb{Z}^d$ [i.e., $\mathbf{P}\{T_x(\cdot)\} = \mathbf{P}\{\cdot\}$, $x \in \mathbb{Z}^d$] such that $v(x, T_y\omega) = v(x+y, \omega)$, $x, y \in \mathbb{Z}^d$. The expectation with respect to the measure $\mathbf{P}$ we will denote by $\mathbf{E}\{\cdot\}$.

We will associate with the given field v the following Cauchy problem:

$$x(t+1) - x(t) = v(x(t)), \qquad x(0) = x_0; \qquad x(t), x_0 \in \mathbb{Z}^d, \quad t \in \mathbb{Z}_+ \tag{1.1}$$

where $\mathbb{Z}_+ = \{0, 1, 2,...\}$, and the relevant random map $V: \mathbb{Z}^d \mapsto \mathbb{Z}^d$ given by

$$V(x) = x + v(x)$$

When t runs overall natural numbers, the solutions $X(t, x_0) = V^t x_0$ of the Cauchy problem generate orbits of the flow associated with the field v. Since the mapping V is not a one-to-one correspondence, we should specify the definition of orbits of the flow. In order to do this, we begin with the following obvious general observation.

Proposition 1. For any map $v: \mathbb{Z}^d \mapsto \mathbb{Z}^d$ any solution $x(t)$ of the Cauchy problem (1.1) must be one of the following two types only:

(i) Non-self-intersecting, i.e., all values $x(t)$, $t \in \mathbb{Z}_+$ are different.

(ii) Periodic beginning from some finite time t_1, i.e., $\exists t_1, T \in \mathbb{Z}_+$: $x(t+T) = x(t)$, $t \geqslant t_1$.

Definition 1. *Orbits of a flow.* Suppose that $\chi = [x_n, x_{n+1},..., x_m]$ is an ordered set in $\mathbb{Z}^d$, where n and m are integers and $n \leqslant m$ (we allow also $n = -\infty$ and $m = \infty$ with the corresponding change of denotation for χ). We suppose also all of $x_n, x_{n+1},...,x_m$ to be different. We will call χ an orbit of the flow V if the following is true: (a) $x_{l+1} = Vx_l$ for any finite integer l such that $n \leqslant l < m$; (b) if n is finite, then $Vx \neq x_n$, $\forall x \in \mathbb{Z}^d$; (c) if m is finite, then $Vx_l \in \{x_n, x_{n+1},..., x_m\}$ [the assumptions (b) and (c) suppose that we have extended the orbit as much as possible]. We will distinguish the following types of orbits:

(i) n is finite, $m = 0$, and $Vx_0 = x_n$; we will call such an orbit of the flow a *cycle*.

(ii) $m = 0$ and $Vx_0 = x_l$, $n < l < 0$; we will call such an orbit of the flow an *extended cycle*.

(iii) $m = \infty$; we will call such an orbit of the flow *regular*.

Definition 2. *Vortex.* We can associate with any fixed cycle c the set w_c of all extended orbits of the flow that terminates by this cycle. Let us call such a set a *vortex*. Thus the set of orbits of the flow is partitioned into the set of regular orbits and set of vortices.

From Proposition 1 and Definitions 1 and 2 we easily obtain the following statement.

Proposition 2. (i) If two orbits of a flow contain a common site, then their parts following this site coincide.

(ii) Two different cycles as well as the different vortices do not involve any common site.

Theorem 3 (Orbits of a flow). With probability 1:

(i) The number of orbits going through any site x of the lattice and involving infinitely many sites before x is finite.

(ii) Any vortex w_c covers a finite domain $\hat{w}_c$ of the lattice.

(iii) $\mathbf{E}\{\sum_c (|\hat{w}_c|/|\hat{c}|)\,|\hat{c} \cap A|\} \leqslant |A|$, where $\hat{c}$ is the set of the sites involved in c, and $|A|$ is the number of sites in a subset A of the lattice.

We can enhance the statements if we assume some sort of independence of $v(x)$ at different sites x.

Assumption I. Let $v(x)$, $x \in \mathbb{Z}^d$, be a random stationary vector field with values in $\mathbb{Z}^d$, and there exist a positive constant $\lambda < 1$ such that for any subset $A \subset \mathbb{Z}^d$

$$\mathbf{P}\{v(x) = \cdot, x \in \mathbb{Z}^d/A \mid v(y) = \cdot\cdot, y \in A\} \geqslant \lambda \mathbf{P}\{v(x) = \cdot, x \in \mathbb{Z}^d/A\} \tag{1.2}$$

Assumption C. Suppose $\mathscr{V}$ is a support of the single distribution of $v(0)$, i.e., $\mathscr{V} = \{v \in \mathbb{Z}^d\colon \mathbf{P}\{v(0) = v\} > 0\}$. There exist the finite sequence $v_1,\ldots,v_k$ such that

$$v_1,\ldots,v_k \in \mathscr{V}, \qquad v_1 + \cdots + v_k = 0 \tag{1.3}$$

Under Assumption I the validity of Assumption C is equivalent to the existence of a cycle with a positive probability.

Now we can formulate the main statement.

Theorem 4. Suppose that the random field v satisfies Assumption I. Then with probability 1 either all solutions of (1.1) are of type (i) from Proposition 1 or all of them are of type (ii) from this proposition. Furthermore:

(i) If Assumption C is valid, then with probability 1 any solution of (1.1) is bounded and consequently periodic beginning from a certain time.

(ii) If Assumption C is not valid, then with probability 1 for any solution $x(t)$ of (1.1) all values $x(t)$, $t \in \mathbb{Z}_+$, are different, and therefore $x(t)$ approaches infinity as t approaches infinity.

Definition 3. Let us denote by $t_0(x_0)$ the minimal value of t such that for some t_1, $X(t, x_0) = X(t_1, x_0)$, where $0 \leqslant t_1 < t$.

It turns out that this moment t_0 is finite with probability one. The next statement indicates some estimation of the distribution of this random moment.

Theorem 5. Under Assumptions I and C there exist such positive constant C and μ that $\mu < 1$, and the following inequality holds:

$$\mathbf{P}\{t_0(x_0) > n\} \leqslant C\mu^n, \qquad n = 1, 2, \ldots; \quad x_0 \in \mathbb{Z}^d \tag{1.4}$$

Theorem 6 (Vortices). Suppose that the random field v satisfies Assumption I. Then, with probability 1 either all orbits of the flow are regular or they form vortices only. Besides, if Assumption C is not valid, then with probability 1 all orbits of the flow are regular. If, on the contrary, Assumption C is valid, i.e., the probability of the existence of a cycle is positive, then the set of orbits is exhausted by vortices only, i.e., $\bigcup_c \hat{w}_c = \mathbb{Z}^d$, where c runs overall cycles. If in addition the field v is bounded by some nonrandom constant, then there exist constants δ and C such that $0 < \delta < 1$, $0 < C < \infty$, and

$$\mathbf{P}\{|\hat{w}_c| \geqslant L^d\} \leqslant C\delta^L \tag{1.5}$$

for any positive integer L.

Thus, under the mentioned conditions, if cycles exist, then complete localization occurs.

2. PROOF OF THE RESULTS FOR A STATIONARY VECTOR FIELD

Bearing in mind the obvious properties of a flow listed in Proposition 2, let us define the order relation in the lattice $\mathbb{Z}^d$ associated with the flow V.

Definition 4. We say that the site x is above y (or y is below x) if for some nonnegative integer m, $y = V^m(x)$. If x is above y and y is above x, we say that x and y are equivalent. If x is above y and y is not above x, we say that x is strictly above y. We call the site x a peak if there are no sites above it.

We define also for x that is above y the function

$$H(x, y) = \min\{m \in \mathbb{Z}: m \geqslant 0,\ V^m x = y\}$$

For any positive integer m and $y \in \mathbb{Z}^d$ we denote $V^{-m}y = \{x \in \mathbb{Z}^d: V^m x = y\}$.

The introduced order relation possesses the following properties, which allow us to distinguish the cycles and vortices from other orbits.

Lemma 1. (i) If c is a cycle and $y \in \hat{c}$, then $\hat{c} = \{V^m y, m = 0, 1,..., |\hat{c}| - 1\}$, and all sites in a cycle are equivalent with respect to the order relation.

(ii) If $m_1 < m_2$ are nonnegative integers, and for some site x, $V^{m_1}x = V^{m_2}x$, then for some cycle c, $V^{m_j}x$, $j = 1, 2$, are in the set $\hat{c}$, and $m_1 \equiv m_2 \pmod{|\hat{c}|}$.

Proof. The statement (i) is the straightforward consequence of the definitions of a cycle and the order relation. Statement (ii) follows from (i) and observation that $y = V^{m_1}x$ is in a cycle, since $V^{m_2 - m_1}y = y$. ▌

Lemma 2. (i) If y is outside any cycle, then for any nonnegative m

$$V^{-m}y = \{x \in \mathbb{Z}^d: H(x, y) = m\} \tag{2.1}$$

and all elements of the set $V^{-(m+1)}y$ (if they exist) are strictly above all elements of the set $V^{-m}y$; in particular, this means that the sets $V^{-m}y$, $m = 0, 1,...,$ are disjoint.

(ii) If for a cycle c, $y \in \hat{c}$, then for any nonnegative integer m

$$V^{-m}y = \{x \in \mathbb{Z}^d: H(x, y) \leqslant m, H(x, y) = m \pmod{|\hat{c}|}\} \tag{2.2}$$

and, for any nonnegative integers k, l, and m_1, m_2 $(m_1 < m_2)$

$$V^{-[k + l|\hat{c}|]}y \subset V^{-[k + (l+1)|\hat{c}|]}y \tag{2.3}$$

$$V^{-m_1}y \cap V^{-m_2}y = \varnothing \qquad \text{if} \quad m_1 \neq m_2 \pmod{|\hat{c}|} \tag{2.4}$$

Proof. To prove (2.1), let us notice that if $H(x, y) = m$, then $V^m x = y$, and therefore $x \in V^{-m}y$. Suppose now that $x \in V^{-m}y$. Then from the definition of $V^{-m}y$ and H we have $V^m x = y$ and $V^{H(x,y)} = y$. If we assume $H(x, y) \neq m$, then from the last equalities and Lemma 1(i) we must conclude that for some cycle c, y is in $\hat{c}$, which contradicts the assumptions of (i). Therefore (2.1) is true. The last part of statement (i) follows from (2.1). To prove (2.2), suppose first that x satisfies the relations in (2.2). Then we have $V^{H(x,y)} = y$, and since $H(x, y) \leqslant m$, $H(x, y) = m \pmod{|\hat{c}|}$, using Lemma 1(i), we may conclude that $V^m x = y$. Now let us assume that $V^m x = y$. Then from this and the definition of H it follows that $H(x, y) \leqslant m$ and $V^{H(x,y)} = y$. From the last equality and Lemma 1(i) we obtain $H(x, y) = m \pmod{|\hat{c}|}$, which completes the proof of (2.2). The relations (2.3), (2.4) easily follow from (2.2). ▌

To establish the main results, let us introduce the following matrix S in $l_2(\mathbb{Z}^d)$ associated with the action of operator V in the lattice $\mathbb{Z}^d$:

$$S(x, y) = \begin{cases} 1 & \text{if} \quad y = V(x), \quad x, y \in \mathbb{Z}^d \\ 0 & \text{otherwise} \end{cases}$$

If we denote by e_y, $y \in \mathbb{Z}^d$, the standard basis in $l_2(\mathbb{Z}^d)$ and by $\mathscr{D}$ the space of finite linear combinations of the basic vectors, we can associate with the matrix S the matrix operator S that acts as follows:

$$S\psi(y) = \sum_x S(y, x)\, \psi(x) = \psi(V(y)), \qquad \psi \in \mathscr{D}, \quad y \in \mathbb{Z}^d$$

or

$$Se_y = \sum_{x:\, V(x) = y} e_x \tag{2.5}$$

If we wish to deal with operator S within the Hilbert space $l_2(\mathbb{Z}^d)$, the sum in (2.5) should involve only a finite number of summands. We can notice here that

$$\|Se_y\|^2 = |V^{-1}y| \tag{2.6}$$

where $\|\cdot\|$ is the norm in $l_2(\mathbb{Z}^d)$. In addition, since we assumed the field v to be stationary, we have

$$S(x, y, T_z\omega) = S(x+z, y+z, \omega), \qquad x, y, z \in \mathbb{Z}^d \tag{2.7}$$

Lemma 3. (i) The following relation holds:

$$\mathbf{E}\{\|Se_y\|^2\} = \mathbf{E}\left\{\sum_x |S(x, y)|^2\right\} = 1, \qquad y \in \mathbb{Z}^d \tag{2.8}$$

and therefore with probability 1, $S\mathscr{D} \subseteq \mathscr{D}$.

(ii) With probability 1, the matrix associated with the mapping V^m, $m = 1, 2,\ldots$, equals S^m, $S^m\mathscr{D} \subseteq \mathscr{D}$, and

$$\mathbf{E}\{\|S^m e_y\|^2\} = 1, \qquad y \in \mathbb{Z}^d, \quad m = 1, 2,\ldots \tag{2.9}$$

$$S^m e_y = \sum_{x:\, V^m(x) = y} e_x, \qquad \|S^m e_y\|^2 = |V^{-m}y| \tag{2.10}$$

Proof. Using the relations (2.5) and (2.7), we have

$$\begin{aligned}\mathbf{E}\{\|Se_y\|^2\} &= \mathbf{E}\left\{\sum_z |S(z, y)|^2\right\} = \sum_z \mathbf{E}\{|S(z, y)|^2\}\\ &= \sum_z \mathbf{E}\{|S(0, y-z)|^2\} = \sum_z \mathbf{E}\{|S(0, z)|^2\}\\ &= \mathbf{E}\{|S(0, V(0))|^2\} = \mathbf{E}\{1\} = 1\end{aligned}$$

Thus, the equality (2.8) is true. Therefore, with probability 1, $S\mathscr{D} \subseteq \mathscr{D}$. Statement (ii) easily follows from (2.6) and (i), which completes the proof of the lemma. ▌

Lemma 4. Let us denote by $v(y)$ the number of infinite orbits of the flow that go through the site y and involve infinitely many sites before y, and by $\mathscr{C}$ the set $\bigcup_c \hat{c}$, where c runs over the set of cycles associated with given flow. Then (i)

$$v(y) \leqslant \liminf_{m \to \infty} \|S^m e_y\|^2 \tag{2.11}$$

and (ii)

$$\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} (e_x, S^k e_y)^2 = \begin{cases} 0 & \text{if } \; y \notin \mathscr{C} \\ 0 & \text{if } \; y \in \hat{c} \subset \mathscr{C}, \quad x \notin \hat{w}_c \\ |\hat{c}|^{-1} & \text{if } \; y \in \hat{c} \subset \mathscr{C}, \quad x \in \hat{w}_c \end{cases} \tag{2.12}$$

Proof. To prove (2.11), let us assume first that $v(y) = v$ is finite. Thus, there exist v different orbits of the flow, each of which involves infinitely many sites of the lattice. Therefore we can find v different sites $x_1,..., x_v$ above y such that each of them belongs to exactly one of v orbits considered. Then, for any $m \geqslant m_0 = \max\{H(x_j, y), 1 \leqslant j \leqslant v\}$ the set $V^{-m}y$ should contain not less than v sites. Indeed, if this is not true, we can find such a $z \in V^{-m}y$ and different x_i and x_l that $x_i = V^{m_i} z$ and $x_l = V^{m_l} z$, where m_i and m_l are nonnegative integers. Supposing, for instance, that $m_i > m_l$, we have $x_i = V^{m_i - m_l} x_l$, which contradicts the assumption that each x_j belongs to exactly one of the considered orbits. That is, $|V^{-m}y| \geqslant v$ for any $m \geqslant m_0$. From this inequality and the equalities (2.10) we obtain the desired (2.11). If $v(y)$ is infinite, we should take any finite v and repeat literally the previous reasoning. Thus we obtain $\liminf(\cdot) \geqslant v$, and since v is arbitrary, we have $\liminf(\cdot) = \infty$, which completes the proof of (2.11).

Considering (2.12), we can notice that, if $y \notin \mathscr{C}$, then, as follows from Lemma 2(i) and (2.10), $(e_x, S^m e_y)$ can be different from zero for one m only. From this it easily follows that (2.12) is true. If $y \in \hat{c}$, then $V^{-m}y \subseteq \hat{w}_c$. Therefore, if $x \notin \hat{w}_c$, using (2.10), we have $(e_x, S^m e_y) \equiv 0$, that is, (2.12) is true. Finally, if $y \in \hat{c}$ and $x \in \hat{w}_c$, we have from (2.10) and Lemma 2(ii) that $(e_x, S^m e_y) = 1$, when $m = H(x, y) (\text{mod } |\hat{c}|)$ and zero otherwise. From this observation we can easily get (2.12), which completes the proof of the lemma. ▌

Lemma 5. (i) We have

$$\mathbf{E}\{v(x)\} \leqslant 1$$

and (ii) for any $\Lambda \subseteq \mathbb{Z}^d$

$$\mathbf{E}\left\{\sum_c \frac{|\hat{w}_c|}{|\hat{c}|} |\hat{c} \cap \Lambda|\right\} \leqslant |\Lambda| \tag{2.13}$$

Proof. Statement (i) follows straightforwardly from (2.9), (2.11), and Fatou's lemma. In order to prove (2.13), let us use (2.12) in the following way. Let us take arbitrary finite subsets Λ and Λ_1 in $\mathbb{Z}^d$, and consider the expectation

$$\mathbf{E}\left\{\sum_{x \in \Lambda_1} \sum_{y \in \Lambda} \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} (e_x, S^k e_y)^2\right\} = \mathbf{E}\left\{\sum_c \frac{|\hat{w}_c \cap \Lambda_1|}{|\hat{c}|} |\hat{c} \cap \Lambda|\right\} \tag{2.14}$$

On the other hand, using Fatou's lemma and (2.9), we can evaluate the left side of (2.14) as follows:

$$\begin{aligned}
&\mathbf{E}\left\{\sum_{x \in \Lambda_1} \sum_{y \in \Lambda} \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} (e_x, S^k e_y)^2\right\} \\
&\quad = \sum_{y \in \Lambda} \mathbf{E}\left\{\sum_{x \in \Lambda_1} \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} (e_x, S^k e_y)^2\right\} \\
&\quad \leqslant \sum_{y \in \Lambda} \mathbf{E}\left\{\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} \|S^k e_y\|^2\right\} \leqslant |\Lambda|
\end{aligned}$$

Thus from the last inequality and (2.14) we have

$$\mathbf{E}\left\{\sum_c \frac{|\hat{w}_c \cap \Lambda_1|}{|\hat{c}|} |\hat{c} \cap \Lambda|\right\} \leqslant |\Lambda|$$

Since in this inequality Λ_1 is an arbitrary set in the lattice, we may conclude that (2.13) is true. ▮

Proof of Theorem 3. Statements (i) and (iii) of the theorem follow in a straightforward way from Lemma 5(i), (ii), respectively. In order to prove statement (ii), let us take an arbitrary sequence of cubes Λ_n, $n \geqslant 1$, expanding to the whole lattice $\mathbb{Z}^d$ when n approaches ∞. Then from (ii) it follows that with probability 1 for any n

$$\sum_c \frac{|\hat{w}_c|}{|\hat{c}|} |\hat{c} \cap \Lambda_n| < \infty$$

Since any cycle is covered by some Λ_n, then all $|\hat{w}_c|$ are finite, which completes the proof of (i) and the theorem. ▮

3. PROOF OF THE RESULTS WHEN THE VECTORS OF THE FLOW ARE INDEPENDENT IN DIFFERENT SITES

We begin the proof of Theorem 4 from statement (ii). Namely, if Assumption C is not valid, then using (1.1), we obtain with probability 1 for any $t > t_1 \in \mathbb{Z}_+$

$$x(t) - x(t_1) = \sum_{\tau = t_1}^{t} v(x(\tau)) \neq 0$$

That is, all $x(t)$, $t \in \mathbb{Z}_+$, are different, which completes the proof of statement (ii) of Theorem 4. Thus, we will assume below that Assumption C is valid.

We will need some auxiliary assertions. Let us introduce the random process $X(t, x_0)$, $t \geqslant 0$, defined by Eq. (1.1) and investigate its properties.

Suppose that $\chi = [x_1,..., x_m]$ is an ordered set in $\mathbb{Z}^d$, i.e., χ is a vector from $(\mathbb{Z}^d)^m$. Given such an χ, we denote by $\hat{\chi}$ the set $\{x_1,..., x_m\}$ corresponding to this χ. Denote by U the operator of the cyclic permutation of any such χ, i.e., $Ux_1 = x_2$, $Ux_2 = x_3,..., Ux_m = x_1$.

We will also use the following notations:

$$Y^l = [y_1,..., y_l] \in (\mathbb{Z}^d)^l, \qquad \hat{Y}^l = \{y_1,..., y_l\}$$

$\mathbf{P}\{\cdot | \cdot\cdot\}$ is the conditioned probabilistic measure, where $\mathbf{P}$ is the original probabilistic measure and the conditions are to the right of the vertical bar. We have

$$p_X(Y^l) = \mathbf{P}\{X(t, x_0) = y_t, 1 \leqslant t \leqslant l\}$$

$$p_X(Z^m | Y^l) = \mathbf{P}\{X(l + \alpha, x_0) = z_\alpha, 1 \leqslant \alpha \leqslant m \,|\, X(\beta, x_0) = y_\beta, 0 \leqslant \beta \leqslant l\}$$

Now we will assume below that k from the Assumption C is the minimal nonnegative integer providing the validity of (1.3). Then, using the corresponding vectors $v_1,..., v_k$ from (1.3), we can construct the ordered set $\chi_0 = [x_1^0,..., x_k^0]$ such that $x_1^0 = 0$, $x_2^0 = v_1,..., x_k^0 = v_1 + \cdots + v_{k-1}$.

Lemma 6. Under the Assumptions I and C:

(i) $x_1^0,..., x_k^0$ are different.

(ii) The probability that χ_0 is a cycle is equal to

$$\mathbf{P}\{V(x_\alpha^0) = x_{\alpha+1}^0, 1 \leqslant \alpha \leqslant k-1, V(x_k^0) = x_1^0\} = p_0 > 0 \qquad (3.1)$$

Proof. Statement (i) follows in a straightforward way from the choice of k as a minimal nonnegative integer providing (1.3). In addition, from the construction of χ_0 we have

$$\mathbf{P}\{V(x_\alpha^0) = x_{\alpha+1}^0, 1 \leqslant \alpha \leqslant k-1, V(x_k^0) = x_1^0\} = \mathbf{P}\{v(x_\alpha^0) = v_\alpha, 1 \leqslant \alpha \leqslant k\}$$

Then, using Assumption I, we can write that

$$\begin{aligned}
&\mathbf{P}\{v(x_\alpha^0)=v_\alpha,\ 1\leqslant\alpha\leqslant k\}\\
&\quad=\mathbf{P}\{v(x_1^0)=v_1\mid v(x_\alpha^0)=v_\alpha,\ 2\leqslant\alpha\leqslant k\}\ \mathbf{P}\{v(x_\alpha^0)=v_\alpha,\ 2\leqslant\alpha\leqslant k\}\\
&\quad\geqslant\lambda\mathbf{P}\{v(x_1^0)=v_1\}\ \mathbf{P}\{v(x_\alpha^0)=v_\alpha,\ 2\leqslant\alpha\leqslant k\}\\
&\quad\geqslant\cdots\geqslant\lambda^k\prod_{1\leqslant\alpha\leqslant k}\mathbf{P}\{v(x_\alpha^0)=v_\alpha\}
\end{aligned}$$

Since the right side of the last inequality is positive (in accordance with the definition of v_α), we have the desired (3.1). ▮

Lemma 7. The conditioned probabilities p_X can be represented as follows:

$$p_X(Z^m\mid Y^l)=\mathbf{P}\{v(z)=Uz-z,\ z\in[y_l,\hat{Z}^m]\backslash z_m\mid v(y)=Uy-y,\ y\in\hat{Y}^l\backslash y_l\} \tag{3.2}$$

Proof. This statement follows at once from the definitions of the random process $X(t, x_0)$ by Eq. (1.1) and the operator of the cyclic permutation U. ▮

Let us denote by $\mathscr{B}_n$, $n\in\mathbb{Z}_+$, such event that all vectors $X(t, x_0)$, $0\leqslant t\leqslant nk$, are different. It is clear that the statements of Theorems 4 and 5 can be reduced to estimates of the probabilities $\mathbf{P}\{\mathscr{B}_n\}$. It is obvious also that

$$\mathscr{B}_{n+1}\subset\mathscr{B}_n,\qquad n\in\mathbb{Z}_+ \tag{3.3}$$

Lemma 8. The following inequality holds:

$$\mathbf{P}\{\mathscr{B}_{n+1}\}\leqslant q_0\mathbf{P}\{\mathscr{B}_n\},\qquad q_0=1-\lambda p_0 \tag{3.4}$$

and consequently

$$\mathbf{P}\{\mathscr{B}_n\}\leqslant q_0^n \tag{3.5}$$

where p_0 is the constant defined in (3.1).

Proof. To obtain the necessary estimations, let us denote by D_n the set of such Y^{nk} that have different components, i.e., $y_t\neq y_\tau$ if $\tau\neq t$ and $1\leqslant t,\ \tau\leqslant nk$. Now we can represent the desired probabilities as follows:

$$P\{\mathscr{B}_n\}=\sum_{Y^{nk}\in D_n}p_X(Y^{nk}) \tag{3.6}$$

$$\begin{aligned}
P\{\mathscr{B}_{n+1}\}&=\sum_{[Y^{nk},Z^k]\in D_{n+1}}p_X(Z^k\mid Y^{nk})\,p_X(Y^{nk})\\
&=\sum_{Y^{nk}\in D_n}p_X(Y^{nk})\sum_{Z^k:\,[Y^{nk},Z^k]\in D_{n+1}}p_X(Z^k\mid Y^{nk})
\end{aligned} \tag{3.7}$$

Let us estimate the sum of the conditioned probabilities in the last equality assuming that $[Y^{nk}, Z^k] \in D_{n+1}$ and using the corresponding representation from Lemma 7,

$$\begin{aligned}
&\sum_{Z^k:[Y^{nk},Z^k]\in D_{n+1}} p_X(Z^k \mid Y^{nk}) \\
&= \sum_{Z^k:[Y^{nk},Z^k]\in D_{n+1}} \mathbf{P}\{v(z) = Uz - z,\ z \in [y_{nk}, Z^k]^\wedge \backslash z_k \mid \\
&\qquad\qquad v(y) = Uy - y,\ y \in \hat{Y}^{nk} \backslash y_{nk}\} \\
&= 1 - \sum_{Z^k:[Y^{nk},Z^k]\notin D_{n+1}} \mathbf{P}\{v(z) = Uz - z, z \in [y_{nk}, Z^k]^\wedge \backslash z_k \mid \\
&\qquad\qquad v(y) = Uy - y,\ y \in \hat{Y}^{nk} \backslash y_{nk}\}
\end{aligned} \tag{3.8}$$

Now let us find an upper estimate for the last sum. This estimate is based in the observation that the considered sum contains a term corresponding to the shifted χ_0. Since χ_0 (or any of its shifts) can form the cycle with probability at least p_0, we will use this fact to get the desired estimate. Namely, considering the mentioned sum under the fixed Y^{nk}, suppose that $Z_0^k = [z_1^0,..., z_k^0] = y_{nk} + \chi_0$. In particular, from the construction of χ_0 we have that all z_j^0 are different, and $z_k^0 = y_{nk} + x_1^0 = y_{nk}$. Therefore there exists an integer r such that $1 \leqslant r \leqslant k$, and $z_1^0,..., z_{r-1}^0 \notin \hat{Y}^{nk}$, $z_r^0 = y_s \in \hat{Y}^{nk}$. Then if r is less than k, we form the periodic sequence $y_s, y_{s+1},..., y_{nk}, z_1^0,..., z_{r-1}^0, z_r^0, y_s, y_{s+1},...,$ and construct the vector Z_1^k, modernizing the vector Z_0^k, as follows. We assign to $z_1^1,..., z_k^1$ the values of the constructed periodic sequence in succession starting from $z_1^1 = z_1^0$. Under all these assumptions, using Assumption I and (3.1), we obtain

$$\begin{aligned}
&\sum_{Z^k:[Y^{nk},Z^k]\notin D_{n+1}} \mathbf{P}\{v(z) = Uz - z,\ z \in [y_{nk}, Z^k]^\wedge \backslash z_k \mid \\
&\qquad\qquad v(y) = Uy - y,\ y \in \hat{Y}^{nk} \backslash y_{nk}\} \\
&\geqslant \mathbf{P}\{v(z) = Uz - z,\ z \in [y_{nk}, Z_1^k]^\wedge \backslash z_k^1 \mid v(y) = Uy - y,\ y \in \hat{Y}^{nk} \backslash y_{nk}\} \\
&= \mathbf{P}\{v(z_\alpha^1) = v_\alpha,\ 1 \leqslant \alpha \leqslant r-1 \mid v(y) = Uy - y,\ y \in \hat{Y}^{nk} \backslash y_{nk}\} \geqslant \lambda p_0
\end{aligned}$$

From the last inequality, the inequality (3.8), and the identities (3.6) and (3.7), we obtain successively (3.3) and (3.4). ∎

Proof of Theorems 4 and 5. Let us assume that for a given realization of the random vector field v and some initial value x_0 the solution of (1.1) is unbounded. In accordance with Proposition 1, this means that the corresponding realization of the random process $X(t, x_0)$ belongs to the set $\mathscr{B} = \bigcap_{n \geqslant 1} \mathscr{B}_n$. But this can happen with probability zero only, since from inequality (3.5) it follows that $\mathbf{P}\{\mathscr{B}\}=0$. Hence with probability one each solution of the system (1.1) is bounded. Moreover, the inequality (1.4) is direct consequence of the inequality (3.5), where $\mu=(\lambda p_0)^{1/k}$, and constants p_0 and k were defined in the proof of Lemma 8. It is clear from the definitions of $t_0(x_0)$ and sets $\mathscr{B}_n$. The last remark completes the proof of the Theorems 4 and 5. ▮

Beginning the proof of Theorem 6, we can notice that all statement of this theorem, except inequality (1.5), are straightforward consequences of Theorems 3 and 4. That is, to complete the proof of Theorem 6, we will suppose that Assumptions I and C are valid, and there exists a positive ρ such that $|v(x)|_0 \leqslant \rho$, $x \in \mathbb{Z}^d$, where for $y=[y_1,..., y_d] \in \mathbb{Z}^d$: $|y|_0 = \max_{1 \leqslant j \leqslant d}\{|y_j|\}$.

Let $\Lambda_R = \{x \in \mathbb{Z}^d: |x|_0 \leqslant R\}$, where R is a positive integer. In accordance with Theorem 4, we can associate with each site x the cycle c_x, by which the solution $X(t, x)$ terminates. Fixing R, let us consider the vortex w_{c_0}. From Definition 2 and Proposition 2 we have the following representation:

$$\hat{w}_{c_0} = \{x \in \mathbb{Z}^d: c_x = c_0\} \tag{3.9}$$

Now let us introduce the events

$$\mathscr{A}_R = \{\omega \in \Omega: \hat{c}_0(\omega) \subseteq \Lambda_R\}, \qquad \mathscr{E}_{x,R} = \{\omega \in \Omega: \hat{c}_x(\omega) \subseteq x + \Lambda_{|x|_0 - R}\}$$

where $x \in \mathbb{Z}^d$, $|x|_0 > 2R$. Since for $|x|_0 > 2R$, $\Lambda_R \cap (x + \Lambda_{|x|_0 - R}) = \varnothing$, we can easily obtain, using (3.9), the following result:

$$\{\hat{w}_{x_0} \subseteq \Lambda_{2R}\} \supseteq \mathscr{A}_R \cap \left(\bigcap_{|x|_0 > 2R} \mathscr{E}_{x,R} \right) \tag{3.10}$$

Denoting by $\mathscr{A}^c$ the complementary event to $\mathscr{A}$, we have the following result.

Lemma 9. (i) $\mathscr{A}_R \supseteq \{t_0(0) < R/\rho\}$, and

$$\mathbf{P}\{\mathscr{A}^c_R\} \leqslant C_1 \mu^{R/\rho} \tag{3.11}$$

(ii) $\mathscr{E}_{x,R} \supseteq \{t_0(x_0) < (|x|_0 - R)/\rho\}$, and

$$\mathbf{P}\{\mathscr{E}^c_{x,R})\} \leqslant C_1 \; \mu^{(|x|_0 - R)/\rho} \tag{3.12}$$

$$\mathbf{P}\left\{\mathscr{A}^c_R \cup \left(\bigcup_{|x|_0 > 2R} \mathscr{E}^c_{x,R} \right)\right\} \leqslant C_2 R^{d-1} \mu^{R/\rho} \tag{3.13}$$

where C_1, C_2 are positive constants, and μ is the constant from (1.4).

Proof. The inclusions (i) and (ii) are straightforward consequences of Definition 3 of the function $t_0(x)$ and the definitions of the relevant sets. The inequalities (3.11) and (3.12) follow from the inequality (1.4) and the inclusions (i) and (ii). Finally, (3.13) follows from (3.11) and (3.12), which completes the proof of the lemma. ∎

Proof of Inequality (1.5). From (3.10) and (3.13) we have for any positive R

$$\mathbf{P}\{\hat{w}_{c0} \subseteq \Lambda_{2R}\} \geqslant 1 - C\delta^R \tag{3.14}$$

where C and δ are positive constants, and δ can be arbitrary but is greater than $\mu^{1/p}$. The desired (1.5) obviously follows from (3.14), which completes the proof of Theorem 6. ∎

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